CompOT

- set of feasible transport plans: $\Gamma(\mu,\nu) := \{\gamma \in \mathbb{R}_+^{M\times N} \mid P_X\gamma = \mu, P_Y\gamma = \nu\}$
- Primal Kantorovich problem: $C(\mu,\nu) := \inf\{\langle c,\gamma\rangle \mid \gamma \in \Gamma(\mu,\nu)\}$
- Dual Kantorovich problem: $C(\mu,\nu) = \sup\{\langle \alpha,\mu\rangle + \langle \beta,\nu\rangle \mid \alpha_i + \beta_j \le c_{i,j}\}$
- $\gamma_{i,j} > 0 \Longrightarrow \alpha_i + \beta_j = c_{i,j}$
- c-transform: $\alpha_j^c := \min_i c_{i,j} - \alpha_i$, β_i^c analog
- Metric
  - $d(x,y) = 0 \Longleftrightarrow x = y$
  - $d(x,y) = d(y,x)$
  - $d(x,y) + d(y,z) \ge d(x,z)$
- Wasserstein distance: $W_{p(\mu,\nu)} := \inf\left\{\sum_{i,j} d(x_i,x_j)^p \gamma_{i,j} \mid \gamma \in \Gamma(\mu,\nu)\right\}^{\frac{1}{p}}$
- Monge property: $c_{i,j} + c_{k,l} \le c_{i,l} + c_{k,j}$ when $i \le k, j \le l$
- monotonous transport plan: $\gamma_{i,j} > 0 \Longrightarrow \gamma_{i',j'} = 0$ for ($i' > i$ and $j' < j$) or ($i' < i$ and $j' > j$)
- if c satisfies the Mongo property and γ is monotonous, then γ is optimal
- Hungarian algorithm, $\mathcal{O}(n^4)$, optimized $\mathcal{O}(n^3)$
  - interest: for each column which row is interested
  - $j* = \arg\min_j c[K,j] - \beta[j]$
  - $\alpha[K] = c[K,j*] - \beta[j*]$
  - while conflict: $(i*,j*) = \arg\min_{i\in I, j\notin J} c[i,j] - \alpha[i] - \beta[j]$
  - $\Delta = c[i*,j*] - \alpha[i*] - \beta[j*]$
  - $\alpha[i] += \Delta, \beta[i] -= \Delta$
  - if new column still free update assignment using interest
- Birkhoff von Neumann: convex hull of permutation matrices = bistochastic matrices
- auction algorithm, $\mathcal{O}\left(M^3 \frac{C}{\varepsilon}\right)$, with epsilon scaling $\mathcal{O}\left(M^4 \log\left(\frac{C}{\varepsilon}\right)\right)$
  - submit bids: find most attractive y, update dual ($\alpha[x] = c[x,y] - \beta[y]$), append to y's bid list
  - accept bids: find best bid ($\arg\min c[\mathrm{bl}[y],y] - \alpha[\mathrm{bl}[y]]$), update assignment and duals ($\beta[y] = c[x,y] - \alpha[x] - \varepsilon$)
- Negative entropy: $H(\gamma) := \sum_{i,j} h(\gamma_{i,j})$ with $h(s) = s\log(s) - s + 1$
- Entropic primal problem: $\min\{\langle c,\gamma\rangle + \varepsilon H(\gamma) \mid \gamma \in \Gamma(\mu,\nu)\}$
- Entropic dual problem: $\sup\langle \alpha,\mu\rangle + \langle \beta,\nu\rangle - \varepsilon \sum_{i,j}\left[\exp\left(-\frac{c_{i,j}-\alpha_i-\beta_j}{\varepsilon}\right) - 1\right]$