

CompOT

- set of feasible transport plans: $\Gamma(\mu, \nu) := \{\gamma \in \mathbb{R}_+^{M \times N} \mid P_X \gamma = \mu, P_Y \gamma = \nu\}$
- Primal Kantorovich problem: $C(\mu, \nu) := \inf\{\langle c, \gamma \rangle \mid \gamma \in \Gamma(\mu, \nu)\}$
- Dual Kantorovich problem: $C(\mu, \nu) = \sup\{\langle \alpha, \mu \rangle + \langle \beta, \nu \rangle \mid \alpha_i + \beta_j \leq c_{i,j}\}$
- $\gamma_{i,j} > 0 \implies \alpha_i + \beta_j = c_{i,j}$
- c-transform: $\alpha_j^c := \min_i c_{i,j} - \alpha_i$, β_i^c analog
- Metric
 - $d(x, y) = 0 \iff x = y$
 - $d(x, y) = d(y, x)$
 - $d(x, y) + d(y, z) \geq d(x, z)$
- Wasserstein distance: $W_{p(\mu, \nu)} := \inf\left\{\sum_{i,j} d(x_i, x_j)^p \gamma_{i,j} \mid \gamma \in \Gamma(\mu, \nu)\right\}^{\frac{1}{p}}$
- Monge property: $c_{i,j} + c_{k,l} \leq c_{i,l} + c_{k,j}$ when $i \leq k, j \leq l$
- monotonous transport plan: $\gamma_{i,j} > 0 \implies \gamma_{i',j'} = 0$ for $(i' > i \text{ and } j' < j)$ or $(i' < i \text{ and } j' > j)$
- if c satisfies the Monge property and γ is monotonous, then γ is optimal
- Hungarian algorithm, $\mathcal{O}(n^4)$, optimized $\mathcal{O}(n^3)$
 - interest: for each column which row is interested
 - $j^* = \arg \min_j c[K, j] - \beta[j]$
 - $\alpha[K] = c[K, j^*] - \beta[j^*]$
 - while conflict: $(i^*, j^*) = \arg \min_{i \in I, j \notin J} c[i, j] - \alpha[i] - \beta[j]$
 - $\Delta = c[i^*, j^*] - \alpha[i^*] - \beta[j^*]$
 - $\alpha[i^*] += \Delta, \beta[j^*] -= \Delta$
 - if new column still free update assignment using interest
- Birkhoff von Neumann: convex hull of permutation matrices = bistochastic matrices
- auction algorithm, $\mathcal{O}(M^3 \frac{C}{\epsilon})$, with epsilon scaling $\mathcal{O}(M^4 \log(\frac{C}{\epsilon}))$
 - submit bids: find most attractive y, update dual ($\alpha[x] = c[x, y] - \beta[y]$), append to y's bid list
 - accept bids: find best bid ($\arg \min c[\text{bl}[y], y] - \alpha[\text{bl}[y]]$), update assignment and duals ($\beta[y] = c[x, y] - \alpha[x] - \epsilon$)
- Negative entropy: $H(\gamma) := \sum_{i,j} h(\gamma_{i,j})$ with $h(s) = s \log(s) - s + 1$
- Entropic primal problem: $\min\{\langle c, \gamma \rangle + \epsilon H(\gamma) \mid \gamma \in \Gamma(\mu, \nu)\}$
- Entropic dual problem: $\sup\langle \alpha, \mu \rangle + \langle \beta, \nu \rangle - \epsilon \sum_{i,j} \left[\exp\left(-\frac{c_{i,j} - \alpha_i - \beta_j}{\epsilon}\right) - 1 \right]$